Chapter 4

Decision Making in a Fuzzy Environment

Decision making is a process of problem solving which results in an action. It is a choice between various ways of getting an end accomplished. Decision making plays an important role in business, finance, management, economics, social and political science, engineering and computer science, biology, and medicine. It is a difficult process due to factors like incomplete and imprecise information, subjectivity, linguistics, which tend to be presented in real-life situations to lesser or greater degree. These factors indicate that a decision-making process takes place in a fuzzy environment. The main objective of this chapter is to consider two methods for decision making based on fuzzy sets and fuzzy logic. First to be introduced is the Bellman–Zadeh (1970) approach, according to which decision making is defined as intersection of goals and constraints described by fuzzy sets. The second approach for making decisions combines goals and constraints using fuzzy averaging. Applications are made to various real-life situations requiring selection or evaluation type decisions and to pricing models. Also a budget allocation procedure is discussed.

4.1 Decision Making by Intersection of Fuzzy Goals and Constraints

Decision making is characterized by *selection* or choice from *alternatives* which are available, i.e. they are found or discovered. In the process of decision making, specified goals have to be reached and specified constraints have to be kept.

Consider a simple decision-making model consisting of a goal described by a fuzzy set $\mathcal{G}$ with membership function $\mu_{\mathcal{G}}(x)$ and a constraint described by a fuzzy set $\mathcal{C}$ with membership function $\mu_{\mathcal{C}}(x)$, where x is an element of the crisp set of alternatives A_{alt}.

By definition (Bellman and Zadeh (1970)) the decision is a fuzzy set $\mathcal{D}$ with membership function $\mu_{\mathcal{D}}(x)$, expressed as intersection of $\mathcal{G}$ and $\mathcal{C}$,

$$\mathcal{D} = \mathcal{G} \cap \mathcal{C} = \{(x, \mu_{\mathcal{D}}(x) | x \in [d_1, d_2], \qquad \mu_{\mathcal{D}}(x) \in [0, h \leq 1]\}. \qquad (4.1)$$

It is a multiple decision resulting in selection the crisp set $[d_1, d_2]$ from the set of alternatives A_{alt}; $\mu_{\mathcal{D}}(x)$ indicates the degree to which any $x \in [d_1, d_2]$ belongs to the decision $\mathcal{D}$. A schematic presentation is shown on Fig. 4.1 when $x \in A_{alt} \subset R$ and $\mathcal{G}$ and $\mathcal{C}$ have monotone continuous membership functions.

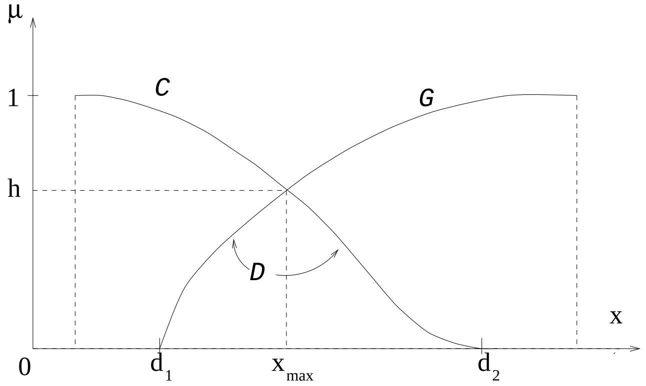


Fig. 4.1. Fuzzy goal $\mathcal{G}$, constraint $\mathcal{C}$, decision $\mathcal{D}$, max decision $x_{\max}$.

Using the membership functions and operation intersection (1.9), formula (4.1) gives

$$\mu_{\mathcal{D}}(x) = \min(\mu_{\mathcal{G}}(x), \mu_{\mathcal{C}}(x)), \qquad x \in A_{alt}. \tag{4.2}$$

The operation intersections is commutative, hence the goal and constraint in (4.1) can be formally interchanged, i.e. $\mathcal{D} = \mathcal{G} \cap \mathcal{C} = \mathcal{C} \cap \mathcal{D}$. Actually there are real situations in which, depending on the point of view, goal could be considered as constraint and vice versa. Sometimes there is no need to specify the goal and constraint; we simply call them objectives or aspects of a problem.

Usually the decision makers want to have a crisp result, a value among the elements of the set $[d_1, d_2] \subset A_{alt}$ which best or adequately represents the fuzzy set $\mathcal{D}$. That requires *defuzzification* of $\mathcal{D}$. It is natural to adopt for that purpose the value x from the selected set $[d_1, d_2]$ with the highest degree of membership in the set $\mathcal{D}$. Such a value x maximizes $\mu_{\mathcal{D}}(x)$ and is called *maximizing decision* (Fig. 4.1). It is expressed by

$$x_{\max} = \{x | \max \mu_{\mathcal{D}}(x) = \max \min(\mu_{\mathcal{G}}(x), \mu_{\mathcal{C}}(x))\}. \tag{4.3}$$

The process of decision making is shown as a block diagram on Fig. 4.2.

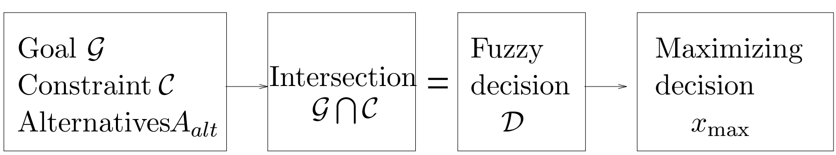


Fig. 4.2. Process of decision making by intersection.

Formulas (4.1)–(4.3) have been generalized for decision-making models with many goals and constraints (Bellman and Zadeh (1970)). For n goals $\mathcal{G}_i, i = 1, \ldots, n$, and m constraints $\mathcal{C}_j, j = 1, \ldots, m$, the decision is

$$\mathcal{D} = \mathcal{G}_1 \cap \cdots \mathcal{G}_n \cap \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_m, \tag{4.4}$$

the membership function of $\mathcal{D}$ is

$$\mu_{\mathcal{D}}(x) = \min(\mu_{\mathcal{G}_1}(x), \ldots, \mu_{\mathcal{G}_m}(x), \mu_{\mathcal{C}_1}(x), \ldots, \mu_{\mathcal{C}_m}(x)), \tag{4.5}$$

and the maximizing decision is given by

$$x_{\max} = \{x|\mu_{\mathcal{D}}(x) \text{ is max}\}. \tag{4.6}$$

If A_{alt} is not a continuous set, for instance a subset of $\mathbf{N}$, the set of integers, formulas (4.1)–(4.6) remain valid.

Example 4.1

On the set of alternatives $A_{alt} = \{1, 2, 3, 4, 5, 6\}$ consider the goal $\mathcal{G}$ and constraint $\mathcal{C}$ given by the discrete fuzzy sets

$$\begin{aligned}\mathcal{G} &= \{(1, 0), (2, 0.2), (3, 0.4), (4, 0.6), (5, 0.8), (6, 1)\},\\ \mathcal{C} &= \{(1, 1), (2, 0.9), (3, 0.7), (4, 0.6), (5, 0.2), (6, 0)\}.\end{aligned}$$

Using the decision formula (4.2) gives (see Fig. 4.3)

$$\begin{aligned}\mathcal{D} = \mathcal{G} \cap \mathcal{C} &= \{(1, \min(0, 1)), (2, \min(0.2, 0.9)), (3, \min(0.4, 0.7)),\\ &\quad (4, \min(0.6, 0.6)), (5, \min(0.8, 0.2)), (6, \min(1, 0))\}\\ &= \{(1, 0), (2, 0.2), (3, 0.4), (4, 0.6), (5, 0.2), (6, 0)\}.\end{aligned}$$

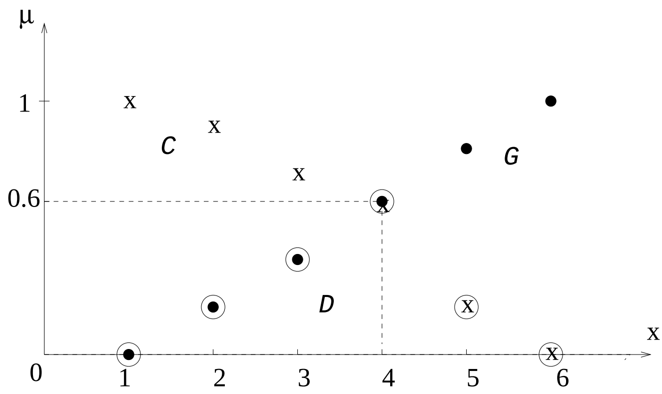


Fig. 4.3. Goal $\mathcal{G}$ (dot), constraint $\mathcal{C}$ (cross), fuzzy decision $\mathcal{D}$ (circle).

Here $[d_1, d_2] = \{1, 2, 3, 4, 5, 6\}, h = 0.6$; the maximizing decision (see (4.3)) is $x_{\max} = 4$ with the highest degree of membership 0.6 in $\mathcal{D}$.

□

We would like to stress that Bellman and Zadeh (1970) made an important comment according to which the definition (4.4) expressing a decision as intersection of goals and constraints is not the only one possible:

"In short, a broad definition of the concept of decision may be stated as *Decision* = Confluence of Goals and Constraints."

Instead of operation intersection (and) defined as *min*, other operations of fuzzy theory could be used to define a decision (see for instance Zimmermann (1984) and Novak (1989)).

We will come back to this point in Section 4.4 where fuzzy averaging is used for decision making.

4.2 Various Applications

Case Study 4 *Dividend Distribution*

In a company the board of directors is willing to pay an attractive dividend to the shareholders but on the other hand, it should be modest. *Attractive dividend*, a linguistic value, is regarded as a goal $\mathcal{G}$ described by a fuzzy set defined on a certain set of alternatives $A_{alt} = \{x|0 < x \leq a\}$, where x is measured in dollars. The membership function $\mu_{\mathcal{G}}(x)$ is increasing on the interval A_{alt}. *Modest dividend* is a constraint $\mathcal{C}$ described by a fuzzy set on A_{alt} with a decreasing membership function $\mu_{\mathcal{C}}(x)$. Good candidate for membership functions are part of triangular or trapezoidal members; also bell-shaped curves could be used.

Assume that the fuzzy set goal $\mathcal{G}$, *attractive dividend*, is defined on the set of alternatives $A_{alt} = \{x|0 < x \leq 8\}$ as

$$\mathcal{G} \stackrel{\triangle}{=} \mu_{\mathcal{G}}(x) = \begin{cases} 0 & \text{for} \quad 0 < x \leq 1, \\ \frac{x-1}{4} & \text{for} \quad 1 \leq x \leq 5, \\ 1 & \text{for} \quad 5 \leq x \leq 8, \end{cases}$$

and the fuzzy set constraint $\mathcal{C}$, *modest dividend*, is given on A_{alt} by

$$\mathcal{C} \stackrel{\triangle}{=} \mu_{\mathcal{C}}(x) = \begin{cases} 1 & \text{for} \quad 0 < x \leq 2, \\ -\frac{x-6}{4} & \text{for} \quad 2 \leq x \leq 6, \\ 0 & \text{for} \quad 6 \leq x \leq 8. \end{cases}$$

According to (4.1) the fuzzy set decision $\mathcal{D}$ is represented by its membership function shown on Fig. 4.4. The crisp set $[d_1, d_2]$ is the interval $[1, 6]$. The intersection point of the straight lines $\mu = \frac{x-1}{4}$ and $\mu = -\frac{x-6}{4}$ is $(3.5, 0.625)$, i.e. $x_{\max} = 3.5, h = \max \mu_{\mathcal{D}}(x) = 0.625$. The dividend to be paid is \$3.5.

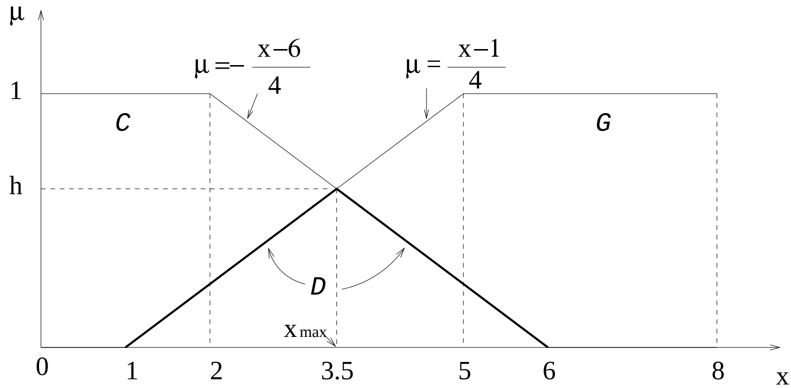


Fig. 4.4. Goal $\mathcal{G}$, constraint $\mathcal{C}$, decision $\mathcal{D}$, maximizing decision $x_{\max}$.

□

Case Study 5 *Job Hiring Policy*

A company advertises a position for which candidates x_k, $k = 1, \ldots, p$, apply; they form the discrete set of alternatives $A_{alt} = \{x_1, \ldots, x_p\}$. The hiring committee requires candidates to possess certain qualities like experience, knowledge in specified areas, etc.[1] which are considered as goals $\mathcal{G}_i, i = 1, \ldots, n$. The committee also wants to impose some constraints $\mathcal{C}_j, j = 1, \ldots, m$, like modest salary, etc.. At the end of the interviewing process each candidate x_k is evaluated from point of view of goals and constraints on a scale from 0 to 1. The score (grade) given to the candidate x_k concerning the goals $\mathcal{G}_i$ is denoted by a_{k_i} and that concerning the constraints $\mathcal{C}$ is denoted by b_{k_j}. Using the scores, committee members construct discrete fuzzy sets $\mathcal{G}_i$ and $\mathcal{C}_j$ on the set of alternatives A_{alt}:

$$\begin{aligned} \mathcal{G}_i &= \{(x_1, a_{1i}), \ldots, (x_p, a_{pi})\}, \quad i = 1, \ldots, n, \\ \mathcal{C}_j &= \{(x_1, b_{1j}), \ldots, (x_p, b_{pj})\}, \quad j = 1, \ldots, m. \end{aligned} \tag{4.7}$$

The decision formula (4.4) gives

$$\mathcal{D} = \mathcal{G}_1 \cap \cdots \mathcal{G}_n \cap \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_m,$$

which with (4.5) produces

$$\mathcal{D} = \{(x_1, \mu_1), \ldots, (x_p, \mu_p)\}, \tag{4.8}$$

where

$$\mu_k = \min(a_{k1}, \ldots, a_{kn}, b_{k1}, \ldots, b_{km}), \quad k = 1, \ldots, p.$$

The candidate with the highest membership grade among $\mu_1, \ldots, \mu_p$ will be considered as the best candidate for the job.

The decision in the numerical Example 4.1 is a particular case of formula (4.8).

Assume that the company wants to fill a position for which there are five candidates $x_i, i = 1, \ldots, 5$, who form the set of alternatives, $\mathcal{A}_{alt} = \{x_1, x_2, x_3, x_4, x_5\}$. The hiring committee has three objectives (goals) which the candidates have to satisfy: (1) experience, (2) computer knowledge, (3) young age. Also the committee has a constraint, the salary offered should be modest. After a serious discussion each candidate is evaluated from point of view of the goals and the constraint. The committee constructs the following fuzzy sets on the set of alternatives (they are a particular case of (4.7) when $n = 3$ and $m = 1$):

$$\begin{aligned}
\mathcal{G}_1 &= \{(x_1, 0.8), (x_2, 0.6), (x_3, 0.3), (x_4, 0.7), x_5, 0.5)\}, \\
\mathcal{G}_2 &= \{(x_1, 0.7), (x_2, 0.6), (x_3, 0.8), (x_4, 0.2), x_5, 0.3)\}, \\
\mathcal{G}_3 &= \{(x_1, 0.7), (x_2, 0.8), (x_3, 0.5), (x_4, 0.5), x_5, 0.4)\}, \\
\mathcal{C} &= \{(x_1, 0.4), (x_2, 0.7), (x_3, 0.6), (x_4, 0.8), x_5, 0.9)\}.
\end{aligned}$$

Here $\mathcal{G}_1$ represents *experience*; $\mathcal{G}_2$, *computer knowledge*; $\mathcal{G}_3$, *young age*; and $\mathcal{C}$ gives the readiness of the candidates to accept a *modest salary*.

The use of the decision formula (4.8) gives

$$\mathcal{D} = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.3), (x_4, 0.2), x_5, 0.3)\}.$$

The candidate x_2 has the largest membership grade 0.6, hence he/she is the best candidate for the job.

The decision model for job hiring, formulas (4.7) and (4.8), can be applied to similar situations framed formally into the same model. The following three case studies fall into that category.

□

Case Study 6 *Selection for Building Construction*

Four buildings are planned for construction consequently in a city, but the order is not determined.[2]

A construction company wants to select the building which will be constructed first. The buildings labeled $b_i, i = 1, \ldots, 4$, form the set of alternatives A_{alt}. The company prefers (has goals) to construct a building which is *not very important* but is *highly profitable* and the construction time is *rather long*. The company is also aware that the city council prefers the first building to be *very important*, with *short construction time*, and *reasonable construction cost*; these are constraints for the company. The management of the company describes the goals and constraints by the following fuzzy sets (b stays for building):

$$
\begin{aligned}
\mathcal{G}_1 &\triangleq \text{not very important } b = \{(b_1, 0), (b_2, 0.4), (b_3, 0.3), (b_4, 0.8)\},\\
\mathcal{G}_2 &\triangleq \text{hightly profitable } b = \{(b_1, 0.5), (b_2, 0.6), (b_3, 0.7), (b_4, 0.3)\},\\
\mathcal{G}_3 &\triangleq \text{long construction time} = \{(b_1, 0.8), (b_2, 0.7), (b_3, 1), (b_4, 0.2)\},\\
\mathcal{C}_1 &\triangleq \text{very important } b = \{(b_1, 1), (b_2, 0.6), (b_3, 0.7), (b_4, 0.2)\},\\
\mathcal{C}_2 &\triangleq \text{short construction time} = \{(b_1, 0.3), (b_2, 0.4), (b_3, 0.5), (b_4, 0.7)\},\\
\mathcal{C}_3 &\triangleq \text{reasonable cost} = \{(b_1, 0.3), (b_2, 0.4), (b_3, 0.7), (b_4, 0.2)\}.
\end{aligned}
$$

The decision according to (4.8) is

$$
\begin{aligned}
\mathcal{D} &= \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3 \cap \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3\\
&= \{(b_1, 0), (b_2, 0.4), (b_3, 0.3), (b_4, 0.2)\}.
\end{aligned}
$$

The company management decision is to propose for construction to the city council the building b_2 with maximum membership value

0.4 in the set $\mathcal{D}$. This decision meets best the goals and constraints. If the proposal is not accepted by the city council, the management is ready to propose for construction building b_3 which is a second choice (membership value 0.3 in $\mathcal{D}$).

Note that $\mathcal{G}_1 \triangleq$ *not very important* b is a complement to $\mathcal{C}_1 \triangleq$ *very important* b, i.e. $\mu_{\mathcal{C}_1}(b) = 1 - \mu_{\mathcal{G}_1}(b)$ (see (1.8)). However, $\mathcal{C}_2 \triangleq$ *short duration* is close but not equal to the complement of $\mathcal{G}_3 \triangleq$ *long duration*, i.e. $\mu_{\mathcal{C}_2}(b) \approx 1 - \mu_{\mathcal{G}_3}(b)$. The linguistic values *short* and *long* are words with opposite meaning and could be described by fuzzy sets which almost complement each other, i.e. *short* $\approx$ *not long*; $\mu_{short}(x) \approx 1 - \mu_{long}(x) = \mu_{notlong}(x)$. However, one has to be careful with the interpretation of words with opposite meaning.

□

Case Study 7 *Housing Policy for Low Income Families*

A city council wants to introduce a housing policy for low income families living in an old apartment building located on a big lot. Three alternative projects are under discussion: p_1 (renovation and housing management), p_2 (ownership transfer program), and p_3 (new construction). The set of alternatives is $A_{alt} = \{p_1, p_2, p_3\}$. Projects p_1 and p_3 will require partial and full relocation of families.

The city council, using the analysis of experts and various interested groups, after long discussions states three goals and one constraint described by fuzzy sets on A_{alt} as follows:

$$\begin{aligned}
\mathcal{G}_1 &\triangleq \textit{improved quality of housing} = \{(p_1, 0.2), (p_2, 0.4), (p_3, 0.8)\},\\
\mathcal{G}_2 &\triangleq \textit{more housing units} = \{(p_1, 0.1), (p_2, 0), (p_3, 0.9)\},\\
\mathcal{G}_3 &\triangleq \textit{better living enviromnent} = \{(p_1, 0.4), (p_2, 0.5), (p_3, 0.8)\},\\
\mathcal{C}_1 &\triangleq \textit{reasonable cost} = \{(p_1, 0.8), (p_2, 0.9), (p_3, 0.4)\}.
\end{aligned}$$

The decision according to (4.8) is

$$\mathcal{D} = \{(p_1, 0.1), (p_2, 0), (p_3, 0.4)\}.$$

Project p_3 with the greatest membership degree 0.4 is preferred over p_1 and p_2; it is superior when goals are concerned, but not that satisfactory as far as cost is concern. □

Case Study 8 *Job Selection Strategy*

A professional person, say Mary, is offered jobs by several companies $c_1, \ldots, c_n$; they form the set of alternatives $A_{alt} = \{c_1, \ldots, c_n\}$. The salaries differ, but Mary while having the goal to earn a high salary, also has in mind certain requirements such as interesting job, job within close driving distance, company with future, opportunity for fast advancement, etc. Those requirements are aspects of the problem and could be considered as constraints (see Section 4.1). Mary expresses the goal of a high salary by a set $\mathcal{G}$ with membership function $\mu_{\mathcal{G}}(x)$ which is continuously increasing in the universal set of salaries located in R_+ measured in dollars. She constructs also the set of constraints on the set of alternatives A_{alt} by attaching to each company a membership value according to her judgement. However the decision making formulas in Section 4.1 are valid for goals and constraints defined on the same set of alternatives. Here the goal is defined on R_+ while the constraints are defined on the set A_{alt} of companies, hence an adjustment is necessary. The set of salaries can be converted to a set located in A_{alt}. For that purpose the salaries $s_1, \ldots, s_n$ offered by companies $c_1, \ldots, c_n$, correspondingly, are substituted into $\mu_{\mathcal{G}}(x)$ and the values $\mu_{\mathcal{G}}(s_1), \ldots, \mu_{\mathcal{G}}(s_m)$, attached to $c_1, \ldots, c_n$, form the set high salary on A_{alt}:

$$\mathcal{G}_{alt} = \{(c_1, \mu_{\mathcal{G}}(s_1)), \ldots, (c_m, \mu(s_m))\}.$$

Assume that Mary must choose one of three jobs[3] offered to her by three different companies c_1, c_2, and c_3; hence the set of alternatives is $A_{alt} = \{c_1, c_2, c_3\}$. The salaries in dollars per year are given on the table

Company	c_1	c_2	c_3
Salary	40,000	35,000	30,000

Mary has the goal to earn a high salary subject to the constraints (aspects): (1) interesting job, (2) job within close driving distance, and (3) company with future. Mary uses her subjective judgement to define the goal and the first two constraints. Regarding the third, she uses her knowledge accumulated by reading the book, *Excelarate: Growing in the New Economy*, by Beck (1995). She describes the constraints by the discrete fuzzy sets

$$\begin{aligned}\mathcal{C}_1 &= \{(c_1, 0.5), (c_2, 0.7), (c_3, 0.8)\},\\ \mathcal{C}_2 &= \{(c_1, 0.3), (c_2, 0.8), (c_3, 1)\},\\ \mathcal{C}_3 &= \{(c_1, 0.3), (c_2, 0.7), (c_3, 0.5)\},\end{aligned}$$

on the set of alternatives (this is the universal set for $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$) and the goal $\mathcal{G}$ of a high salary by the continuous membership function

$$\mathcal{G} \triangleq \mu_{\mathcal{G}}(x) = \begin{cases} 0 & \text{for} \quad 0 < x < 25000, \\ \frac{x-25000}{20000} & \text{for} \quad 25000 \le x \le 45000, \\ 1 & \text{for} \quad 45000 \le x \end{cases}$$

on the universal set R_+ of salaries (see Fig. 4.5).

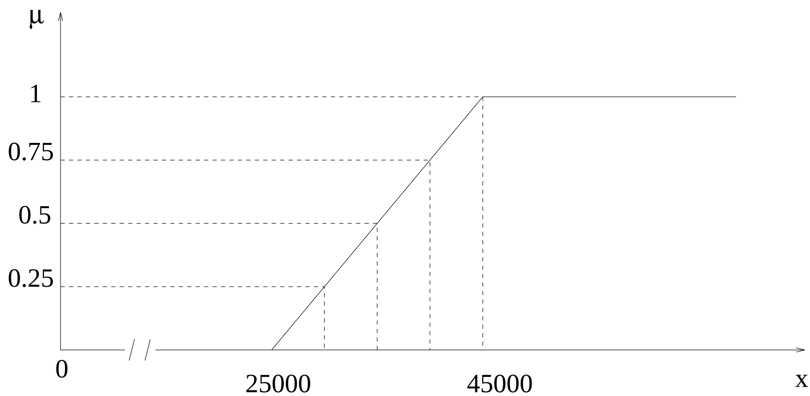


Fig. 4.5. Goal $\mathcal{G}$—high salary.

In order to apply a decision-making formula of the type (4.4) Mary has to deal with one universal set, that of the alternatives. For that purpose she generates membership values by substituting in $\mu_{\mathcal{G}}(x)$, for x, the salaries corresponding to the alternatives,

$$\mu_{\mathcal{G}}(40,000) = 0.75, \qquad \mu_{\mathcal{G}}(35,000) = 0.5, \qquad \mu_{\mathcal{G}}(30,000) = 0.25.$$

As a consequence, the fuzzy set goal $\mathcal{G}$ on the universe R_+ is now substituted by the fuzzy set goal $\mathcal{G}_{alt}$ on the set of the alternatives,

$$\mathcal{G}_{alt} = \{(c_1, 0.75), (c_2, 0.5), (c_3, 0.25)\}.$$

The decision is then (see (4.4))

$$\mathcal{D} = \mathcal{G}_{alt} \cap \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 = \{(c_1, 0.3), (c_2, 0.5), (c_3, 0.25)\}.$$

The maximum membership value in $\mathcal{D}$ is 0.5, hence Mary has to take the job with company c_2 if she wants to satisfy best her objectives. □

Case Study 9 *Evaluation of Learning Performance*[4]

The management of a company established an annual university undergraduate scholarship to support a high school student with excellent performance in science (Mathematics, Physics, Chemistry) and in English. *Excellent* is a linguistic label which the management described separately for science (**ES**) and English (**EE**) on Fig. 4.6 (a) and (b), correspondingly, using part of trapezoidal numbers on the universe $[0, 100]$ of scores.

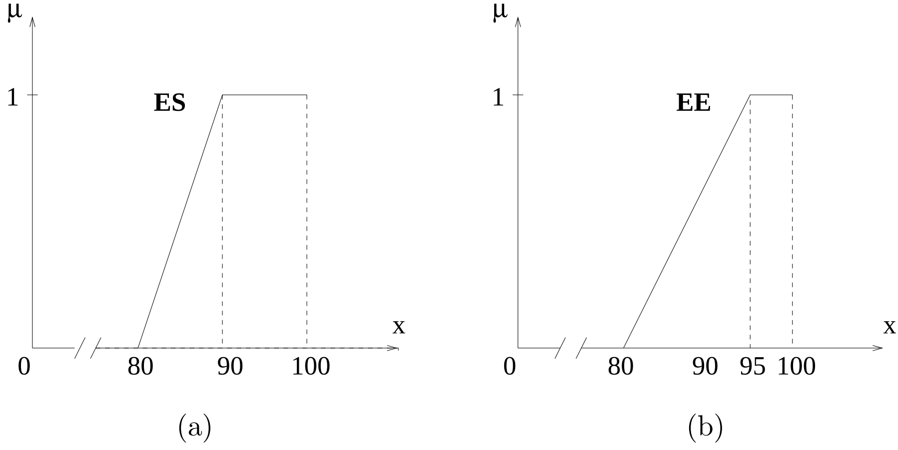


Fig. 4.6. (a) *Excellent* in Science; (b) *Excellent* in English.

The using of (1.15) gives the membership functions

$$\mathbf{ES} \triangleq \mu_{\mathbf{ES}}(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq 80, \\ \frac{x-80}{10} & \text{for} \quad 80 \leq x \leq 90, \\ 1 & \text{for} \quad 90 \leq x \leq 100; \end{cases} \tag{4.9}$$

$$\mathbf{EE} \triangleq \mu_{\mathbf{EE}}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le 80, \\ \frac{x-80}{15} & \text{for} \quad 80 \le x \le 95, \\ 1 & \text{for} \quad 95 \le x \le 100. \end{cases} \tag{4.10}$$

A student's score of 90 in Science has grade of membership 1 in the set **ES** while the same score in English has grade of membership of only 0.67 in the set **EE**.

Five students are candidates for the scholarship, $x_1 =$ Henry, $x_2 =$ Lucy, $x_3 =$ John, $x_4 =$ George, $x_5 =$ Mary. The students' scores are presented in the table bellow.

Table 4.1. Students' scores in Science and English.

	Mathematics	*Physics*	*Chemistry*	*English*
$Henry(x_1)$	86	91	95	93
$Lucy(x_2)$	98	89	93	90
$John(x_3)$	90	92	96	88
$George(x_4)$	96	90	88	89
$Mary(x_5)$	90	87	92	94

The set of alternatives is $A_{alt} = \{x_1, x_2, x_3, x_4, x_5\}$.

Substituting the students scores in Mathematics, Physics, Chemistry into (4.9) and those in English into (4.10) gives the degrees of excellence corresponding to the scores. They are shown on Table 4.2.

Table 4.2. Students' degrees of excellence in Science and English.

	Mathematics	*Physics*	*Chemistry*	*English*
$Henry(x_1)$	0.6	1	1	0.87
$Lucy(x_2)$	1	0.9	1	0.67
$John(x_3)$	1	1	1	0.53
$George(x_4)$	1	1	0.8	0.60
$Mary(x_5)$	1	0.7	1	0.93

The degrees of excellence, attached to each student, produce the fuzzy sets of excellence in Science and English which form the objectives or aspects of the problem:

$$\begin{aligned} &\textit{Excellent in Mathematics} \triangleq \mathcal{G}_1 \\ &= \{(x_1, 0.6), (x_2, 1), (x_3, 1), (x_4, 1), (x_5, 1)\}, \end{aligned}$$

$$\begin{aligned}&\textit{Excellent in Physics} \stackrel{\triangle}{=} \mathcal{G}_2\\&= \{(x_1, 1), (x_2, 0.9), (x_3, 1), (x_4, 1), (x_5, 0.7)\},\\&\textit{Excellent in Chemistry} \stackrel{\triangle}{=} \mathcal{G}_3\\&= \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0.8), (x_5, 1)\},\\&\textit{Excellent in English} \stackrel{\triangle}{=} \mathcal{G}_4\\&= \{(x_1, 0.87), (x_2, 0.67), (x_3, 0.53), (x_4, 0.6), (x_5, 0.93)\}.\end{aligned}$$

The decision formula (4.4) gives

$$\begin{aligned}\mathcal{D} &= \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3 \cap \mathcal{G}_4\\&= \{(x_1, 0.6), (x_2, 0.67), (x_3, 0.53), (x_4, 0.6), (x_5, 0.7)\},\end{aligned}$$

hence the conclusion is that x_5, i.e. Mary with the degree of membership 0.7 in $\mathcal{D}$ is the student with the best performance.

Similar approach could be used to evaluate different types of employee performance in a company or industry.

□

4.3 Pricing Models for New Products

Pricing a new product by a company is a complicated task. It requires the combined efforts of financial, marketing, sales, and management experts to recommend the initial price of a new consumer product. It is also a responsible task since overpricing could create a market for the competitor.

Here we develop a pricing model using the decision method in Section 4.1. The model is based on requirements R_i (rules or objectives) designed by experts. Below are listed some typical requirements[5]:

$R_1 \stackrel{\triangle}{=}$ The product should have *low price*;

$R_2 \stackrel{\triangle}{=}$ The product should have *high price*;

$R_3 \stackrel{\triangle}{=}$ The product should have *close price to double manufacturing cost*; (4.11)

$R_4 \stackrel{\triangle}{=}$ The product should have *close price to competition price*;

More requirements or rules relevant to a particular situation could be added. For instance,

$$R_5 \triangleq \text{The product should have } \textit{slightly higher price} \text{ } \textit{than the competition price.}$$

The linguistic values *low price, high price, close price* can be modified by the modifiers *very* and *fairly* (Section 2.3) which leads to modified requirements.

A particular pricing model should contain at least two requirements.

Considering the requirements as objectives or aspects of a problem the decision-making procedure in Section 4.1 can be applied without any need to specify goals and constraints.

The conflicting linguistic values *low price* and *high price* can be described by right and left triangular or trapezoidal numbers on the set of alternatives, a subset of R_+, measured in dollars. The linguistic value *close price* can be described by triangular numbers. We denote the fuzzy number describing the linguistic value in requirement R_i by $\mathbf{A}_i$. To show the use of pricing requirements in establishing pricing policy we discuss three closely related models.

Case Study 10 *Pricing Models with Three Rules*

Model 1. Consider a pricing model consisting of the three rules (requirements) R_1, R_3 and R_4 stated in (4.11). Assume that the competition price is 25 and the double manufacturing cost is 30. Assume also that the set of alternatives A_{alt} is the interval [10, 50], meaning that the price of the product should be selected from the numbers in this interval.

The model is shown on Fig. 4.7. The linguistic values in the rules are described by fuzzy numbers as follows: R_1 is represented by the right triangular number $\mathbf{A}_1$ (*low price*), R_3 and R_4 are represented by the triangular numbers $\mathbf{A}_3$ (*close to competition price*) and $\mathbf{A}_4$ (*close to double manufacturing cost*), correspondingly.

The analytical expressions of $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{A}_3$ are

$$\mathbf{A}_1 \triangleq \mu_{\mathbf{A}_1}(x) = \begin{cases} \frac{-x+40}{30} & \text{for} \quad 10 \le x \le 40, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{A}_3 \triangleq \mu_{\mathbf{A}_3}(x) = \begin{cases} \frac{x-20}{5} & \text{for} \quad 20 \le x \le 25, \\ \frac{-x+30}{5} & \text{for} \quad 25 \le x \le 30, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{A}_4 \triangleq \mu_{\mathbf{A}_4}(x) = \begin{cases} \frac{x-25}{5} & \text{for} \quad 25 \le x \le 30, \\ \frac{-x+35}{5} & \text{for} \quad 30 \le x \le 35, \\ 0 & \text{otherwise.} \end{cases}$$

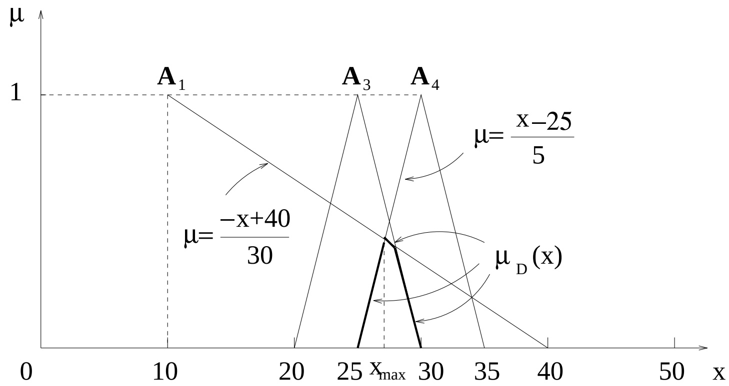


Fig. 4.7. Pricing model with rules R_1, R_3, R_4.

Using (4.5) gives the decision $\mathcal{D}$ (Fig. 4.7) in the interval $[25, 30]$,

$$\mathcal{D} \triangleq \mu_{\mathcal{D}}(x) = \min(\mu_{\mathbf{A}_1}(x), \mu_{\mathbf{A}_3}(x), \mu_{\mathbf{A}_4}(x)).$$

Solving together $\mu = \frac{-x+40}{30}$ and $\mu = \frac{x-25}{5}$ gives the maximizing decision

$$x_{\max} = 27.14,$$

interpreted as price for the product. The experts accept this price as a recommendation. For instance, 14 cents in the price is not customary. The experts may consider a price close to 27.14 in the interval [25, 30], say 27, 26.95, or 26.99.

One can observe from Fig. 4.7 that the triangular number $\mathbf{A}_3$ (*close to competition price*) contributes to the fuzzy decision $\mathcal{D}$, but does not have any impact on the maximizing decision $x_{\max}$. Only the triangular numbers $\mathbf{A}_4$ (*close to double manufacturing cost*) and $\mathbf{A}_1$ (*low price*) contribute to $x_{\max}$. A major role is played by $\mathbf{A}_4$ whose peak with maximum membership grade 1 is at $x = 30$, the double manufacturing cost. Due to the influence of $\mathbf{A}_1$ the maximizing price is 27.14.

Model 2. Now we study the pricing Model 1 when the requirement R_1 defined by $\mathbf{A}_1$ is modified by the modifiers: (a) *very*; (b) *fairly*.

(a) The modified R_1 by *very* reads

$$very R_1 \triangleq \text{The product should have } \textit{very low price}.$$

According to (2.6) the membership function of *very* $\mathbf{A}_1$ is

$$\mu_{very\mathbf{A}_1}(x) = (\mu_{\mathcal{A}_1}(x))^2 = \begin{cases} (\frac{-x+40}{30})^2 & \text{for} \qquad 10 \leq x \leq 40, \\ 0 & \text{otherwise} \qquad . \end{cases}$$

It is a parabola in the interval [10, 40] (Fig. 4.8).

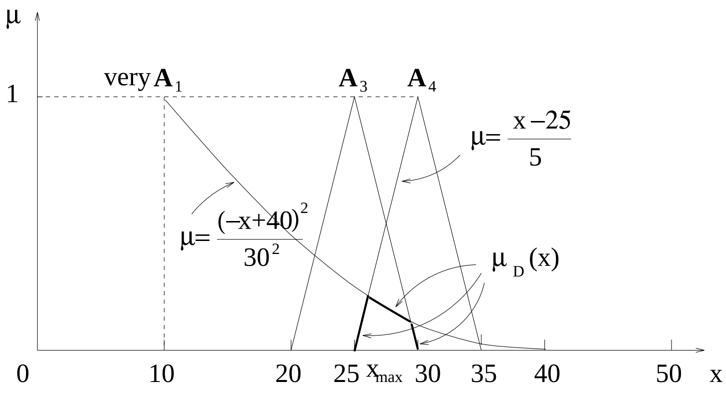


Fig. 4.8. Pricing model with rules *very* R_1, R_3, R_4.

The decision $\mathcal{D}$ has a membership function $\mu_{\mathcal{D}}(x)$ in the interval $[25, 30]$ (Fig. 4.8),

$$\mu_{\mathcal{D}}(x) = \min(\mu_{very\mathbf{A}_1}(x), \mu_{\mathbf{A}_3}(x), \mu_{\mathbf{A}_4}(x)).$$

To find $x_{\max}$ here we have to solve together $\mu = (\frac{-x+40}{30})^2$ and $\mu = \frac{x-25}{5}$ which gives the quadratic equation $x^2 - 260x + 6100 = 0$ with solutions 26.08 and 233.92. The solution in [25, 30], i.e. $x_{\max} = 26.08 \approx 26$, gives the suggested product price.

The modifier *very* gives more emphasis on *low price*. That is why here we get 26, a smaller price than 27.14 obtained in Model 1 (although both models have the same domain).

Here, similarly to Model 1, $\mathbf{A}_3$ (*close to competition price*) contributes to the fuzzy decision $\mathcal{D}$ but not to the maximizing decision.

(b) The modified R_1 by *fairly* reads

$$fairly\ R_1 \triangleq \text{The product should have } \textit{fairly low price.}$$

Using (2.7) gives the membership function of *fairly* A_1.

$$\mu_{fairly}\mathbf{A}_1 = (\mu_{\mathbf{A}_1}(x))^{\frac{1}{2}} = \begin{cases} (\frac{-x+40}{30})^{\frac{1}{2}} & \text{for} \quad 10 \leq x \leq 40, \\ 0 & \text{otherwise} \end{cases}$$

which is a parabola in the interval [10, 40] (Fig. 4.9).

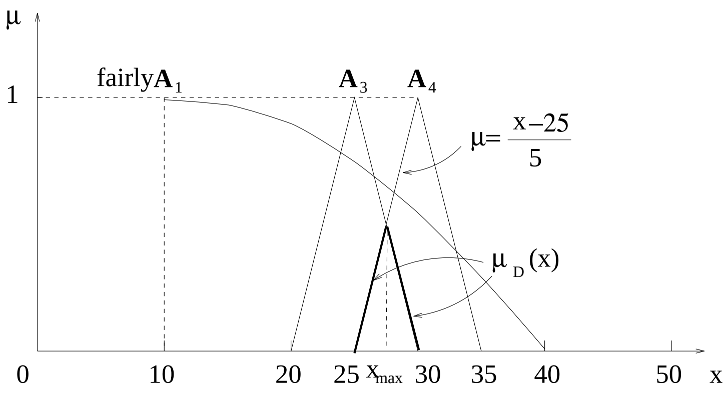


Fig. 4.9. Pricing model with rules *fairly* R_1, R_3, R_4.

From the figure is clear that the rule *fairly* R_1 (fairly *low price*) does not contribute to the fuzzy decision $\mathcal{D}$ with membership function $\mu_{\mathcal{D}}(x)$ on the domain $D = [25, 30]$. Only the rules R_3 and R_4, i.e. $\mathbf{A}_3$ and $\mathbf{A}_4$ contribute to $\mathcal{D}$. The maximizing decision is the midpoint of [25, 30], $x_{\max} = 27.5$.

□

Pricing models like Model 1 and Model 2(a) in Case Study 10 produce maximizing decisions based on *low price* and *doubled manufacturing cost* without reflecting the *competition price* which takes part in the model.[6] A company with such product pricing policy may create favorable market conditions for the competitor. As a consequence the company may incur loses leading to actions as price cutting, redesigning the product, or dropping it from the market. Real-life examples (*Managing in a Time of Great Change*, Drucker[7] (1995)) tell us that it may be more important for a company to consider seriously competition price than to try to make a quick profit of premium pricing. "The only sound way to price is to start out with what the market is willing to pay—and thus, it must be assumed, what competition will charge—and design to that price specification." The next model illustrates Drucker's suggestion: "price-led costing."

Cast Study 11 *A Price-Led Costing Model*

A simple model to reflect "price-led costing" consists of two rules, R_1 (*low price*) and R_3 (*close to competition price*) (see (4.11)). Assume R_1 and R_3 are described by the triangular numbers $\mathbf{A}_1$ and $\mathbf{A}_3$ defined in Model 1 (Case Study 10); they are shown in Fig. 4.10.

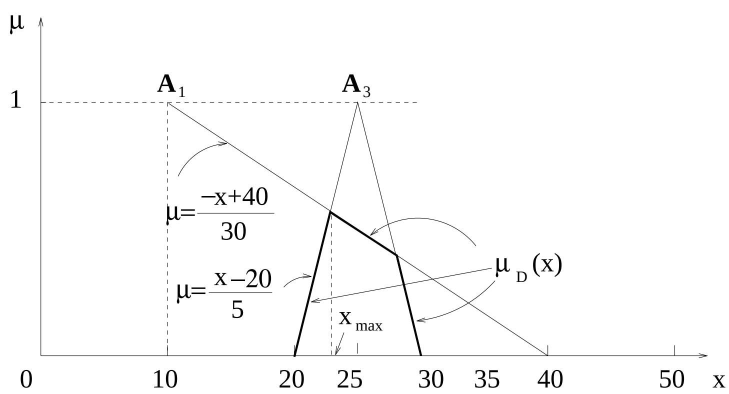


Fig. 4.10. A price-led costing model with rules R_1 and R_3.

The fuzzy decision $\mathcal{D}$ on the domain $\mathcal{D} = [20, 30]$ is

$$\mathcal{D} \triangleq \mu_{\mathcal{D}}(x) = \min(\mu_{\mathbf{A}_1}(x), \mu_{\mathbf{A}_3}(x)).$$

The maximizing decision in [20, 30] is the solution of equations $\mu = \frac{x-20}{5}$ and $\mu = \frac{-x+40}{30}$; it is $x_{\max} = 22.66$, below the competition price of 25 due to the requirement *low price*.

This pricing model, contrary to the models in Case Study 10, does not include a requirement concerning manufacturing cost. The price 22.66 should be considered as a suggestion. The product should be designed, produced, and marketed at cost to ensure that profit could be made if the price of the product is 22.66 or close to it.

□

If a product is new on the market and there is no competition then a reasonable price which consumers are willing to pay should be suggested. A possible model can be based on rules R_1, R_2, and R_4 in (4.11).

If a product is superior to the product of competition then this should be reflected in the model by including rule R_5. A more sophisticated and general model could contain instead of R_5 rules of the type:

"If the product is *superior to the product of competition*, the product price should be *higher than that of competition*."

This is a conditional statement (Chapter 2, Section 2.3). Models with if ... then rules are discussed in Chapter 6.

We have seen that in some pricing models (Case Study 10) there are rules which do not contribute to the decision. The root of the problem lies in the decision-making procedure based on intersection. Formula (4.3) does not always assure contribution from all rules that participate in the model. In those cases decision making by intersection may not be the most appropriate technique to be used. Another approach towards decision making which takes contribution from all goals and constraints (or rules) is based on fuzzy averaging. It is presented in the next section.

4.4 Fuzzy Averaging for Decision Making

In this section the fuzzy averaging technique (Chapter 3, Section 3.1) is used for making decisions. Goals and constraints, or requirements (rules) are described by triangular or trapezoidal numbers. If they are ranked according to importance, the weighted fuzzy averaging is applied. The result (conclusion, aggregation) is a triangular or trapezoidal number **D** interpreted as *decision*. We call this approach *averaging decision making*. To find a maximizing decision we consider the value in the supporting interval of **D** for which $\mu_{\mathbf{D}}(x)$ has maximum membership degree (it is one)(see (3.15) and (3.17)). Also the statistical averages (3.16) and (3.18) could be used.

Case Study 12 *Dividend Distribution by Fuzzy Averaging and Weighted Fuzzy Averaging*

1. Let us apply the fuzzy averaging technique for the problem discussed in Case Study 4 (Section 4.1). The goal **G** (*attractive dividend*) and the constraint **C** (*modest dividend*) (Fig. 4.3 and Fig. 4.11) are right and left trapezoidal numbers. They can be presented as (see Section 1.6)

$$\mathbf{G} = (1, 5, 8, 8), \qquad \mathbf{C} = (0, 0, 2, 6).$$

Using direct calculations (or the trapezoidal average formula (3.13)) gives the trapezoidal number

$$\begin{aligned} \mathbf{D} = \mathbf{A}_{ave} &= \frac{\mathbf{G} + \mathbf{C}}{2} = \frac{(1, 5, 8, 8) + (0, 0, 2, 6)}{2} \\ &= \frac{(1, 5, 10, 14)}{2} = (0.5, 2.5, 5, 7) \end{aligned}$$

which represents the decision (see Fig. 4.11).

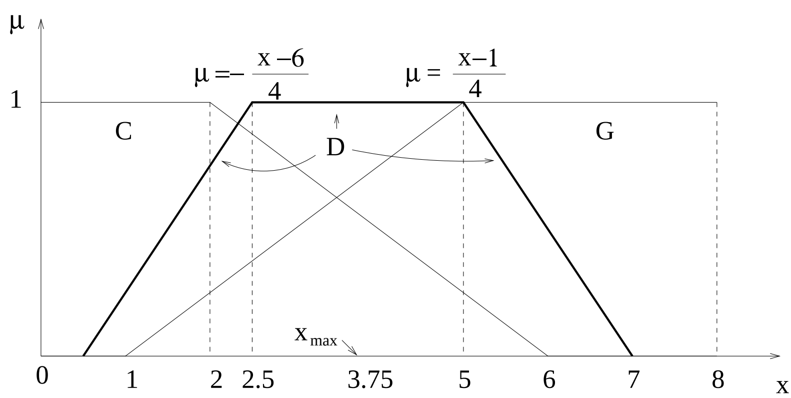


Fig. 4.11. Decision $\mathbf{D}, x_{\max} = 3.75$.

The membership function $\mu_{\mathbf{D}}(x)$ of the decision has a flat segment whose projection on x-axis is the interval [2.5, 5]. The numbers in this interval have the highest degree of membership in $\mathbf{D}$. We define the maximizing decision as the midpoint of the flat interval (see (3.17)), i.e.

$$x_{\max} = \frac{2.5+5}{2} = \frac{7.5}{2} = 3.75.$$

The maximizing decision obtained in Case Study 4 by the intersection method is 3.5. It is up to the board of directors to decide which value to take.

2. Assume now that the board of directors gives different weight to $\mathbf{G}$ and $\mathbf{C}$, for instance $w_{\mathbf{G}} = 0.4$ and $w_{\mathbf{C}} = 0.6$, meaning that the constraint (*modest dividend*) is a little more important than the goal (*attractive dividend*). Then following (3.14) gives the decision

$$\begin{aligned}
\mathbf{D} = \mathbf{A}^w_{ave} &= (0.4)\mathbf{G} + (0.6)\mathbf{C} \\
&= (0.4)(1,5,8,8) + (0.6)(0,0,2,6) \\
&= (0.4,2,3.2,3.2) + (0,0,1.2,3,6) \\
&= (0.4,2,4.4,6.8)
\end{aligned}$$

expressed as a trapezoidal number with a flat interval [2, 4.4]. The midpoint of the flat (formula (3.17)) gives the maximizing decision

$$x_{\max} = \frac{2+4.4}{2} = \frac{6.4}{2} = 3.2$$

which as expected is smaller than 3.75, the case without preference.

Case Study 13 *Two Pricing Models*

Model 1. Consider the pricing Model 1 (Case Study 10) presented on Fig. 4.7 and again on Fig. 4.12. The rules R_1, R_3, and R_4 are described by triangular numbers which can be written in the form of

$$\mathbf{A}_1 = (10,10,40), \qquad \mathbf{A}_3 = (20,25,30), \qquad \mathbf{A}_4 = (25,30,35).$$

Using the triangular average formula (3.13) or direct calculations one gets the decision

$$\mathbf{D} = \mathbf{A}_{ave} = \frac{\mathbf{A}_1 + \mathbf{A}_3 + \mathbf{A}_4}{3}$$

$$= \frac{(10,10,40)+(20,25,30)+(25,30,35)}{3}$$
$$= \frac{(55,65,105)}{3}$$
$$= (18.33, 21.67, 35).$$

It is a triangular number shown in Fig. 4.12.

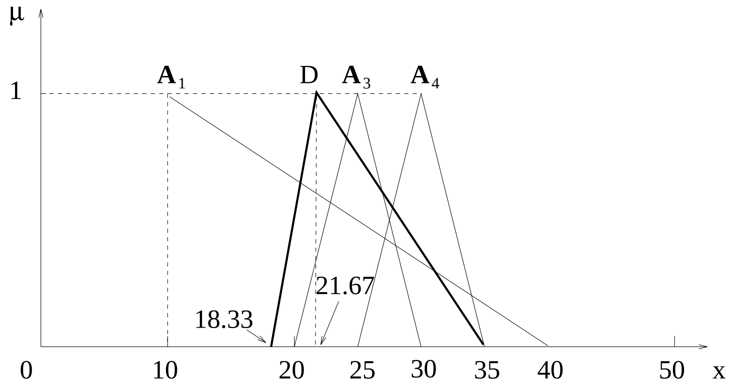


Fig. 4.12. Pricing model with rules R_1, R_3, R_4.

The maximizing decision according to (3.15) is $x_{\max} = 21.67$ since at this value the membership function $\mu_{\mathbf{D}}(x)$ is maximum. The maximizing decision for Model 1, Case Study 10, is 27.14. The difference between the two decisions is not small. Then which value is the correct one? There is no definitive answer to this question. Both decisions should be considered as suggestions. The experts have to make a final decision. The value 27.14 is too high; it does not reflect competition price presented by $\mathbf{A}_3$. On the other hand side, the value 21.67 looks too small; it is not around $\mathbf{A}_4$ although it is influenced by it. A compromise could be to take the number (average) between 21.67 and 27.14 which is $24.405 \approx 24.4$.

Model 2. Let us describe rule R_1 in Model 1 in a slightly different way; the rest remains unchanged. The new right triangular number is $\mathbf{A}_1 = (10, 10, 25)$ (see Fig. 4.13); it has the same peak 1 as the old A_1.

Using the new $\mathbf{A}_1$, and $\mathbf{A}_3$ and $\mathbf{A}_4$ from Model 1, the triangular averaging gives

$$\begin{aligned}\mathbf{D} = \mathbf{A}_{ave} &= \frac{(10, 10, 25) + (20, 25, 30) + (25, 30, 35)}{3} \\ &= \frac{(55, 65, 90)}{3} \\ &= (18.33, 21.67, 30).\end{aligned}$$

It is a triangular number shown on Fig. 4.13. The maximizing decision is $x_{\max} = 21.67$; the same as in Model 1.

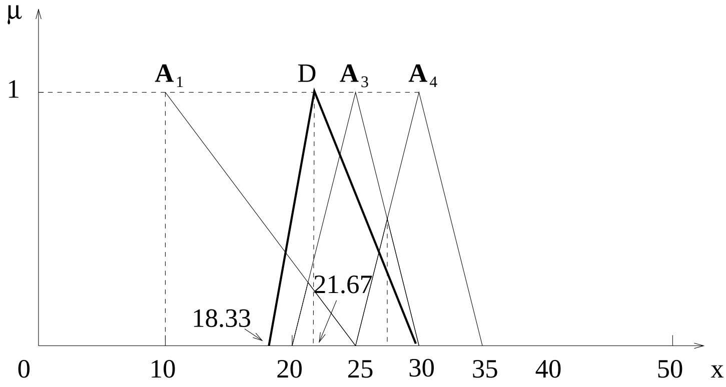


Fig. 4.13. Pricing model with rules R_3, R_4, and slightly different R_1.

Just to make a comparison, let us apply the decision-intersection method to the same model. Noticing that $\mathbf{A}_1$ intersects $\mathbf{A}_3$ but not $\mathbf{A}_4$ above the x-axis, the decision $\mathcal{D}$,

$$\mathcal{D} \triangleq \mu_{\mathcal{D}} = \min(\mu_{\mathbf{A}_1}(x), \mu_{\mathbf{A}_3}(x), \mu_{\mathbf{A}_4}(x)),$$

which is supposed to be a fuzzy set, degenerates into the point (25,0). Recall that when performing operation min the smallest value of μ for each x takes part in $\mathcal{D}$. The number 25 looks like a maximizing decision, but since its degree of membership is zero, the decision intersection method is not the proper tool to be used in this case.

4.5 Multi-Expert Decision Making

Analysis of complex problems requires the efforts and opinions of many experts. Expert opinions are expressed by words from a natural and professional language. They can be considered as linguistic values, hence described and handled by fuzzy sets and fuzzy logic.

It is unlikely that expert opinions are identical. Usually they are either close or conflicting to various degrees. They have to be combined or reconciled in order to produce one decision. We call this multi-expert decision-making procedure *aggregatoin*; it is a conflict resolution when the opinions are confliction. The aggregation is obtained by applying the fuzzy averaging (Section 3.3). It is illustrated on two case studies concerning individual investment planning policy proposed by experts whose opinions are in the first case close and in the second case conflit-ing.

Case Study 14 *Investment Model Under Close Experts Opinions*

Consider a simplified individual investment planning model that produces an *aggressive* or *conservative* policy depending on wheter the interest rates are fallign or rising (see Cox (1995)).

The words *aggresive* and *conservative* are linguistic variables, i.e. fuzzy concepts. The financial experts dealing with the investment model agree to describe *aggressive* (aggressive investment policy) by a suitable left trapezoidal number on a scale from 0 to 100 (universal set – the interval [0, 100]) and *conservative* by a right trapezoidal number on a scale from -100 to 0 (universal set $[-100, 0]$). The numbers on the joined scale from -100 to 100 have a certain meaning accepted by the experts. For instance 50 and -50 can be interpreted as indicators for moderately aggressive investment and moderately conservative investment, correspondingly; 70 and -70 as aggressive and conservative investments, etc.

Assume that interest rates are falling and three experts $E_i, i = 1, 2, 3,$ have the opinion that the investment policy should be agreessive. Their description of *aggressive* is given in the form of left trapezoidal numbers (see Fig. 4.14)

$$\mathbf{A}_1 = (40, 70, 100, 100),\ \ \mathbf{A}_2 = (45, 80, 100, 100),\ \ \mathbf{A}_3 = (70, 85, 100, 100).$$

The aggregation of the close experts opinions (assumed of equal importance) according to the trapezoidal average formula (3.13) produces

$$
\begin{aligned}
\mathbf{A}_{ave} &= \frac{\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3}{3} \\
&= \frac{(40, 70, 100, 100) + (45, 80, 100, 100) + (70, 85, 100, 100)}{3} \\
&= \frac{(155, 235, 300, 300)}{3} = (51.66, 78.33, 100, 100).
\end{aligned}
$$

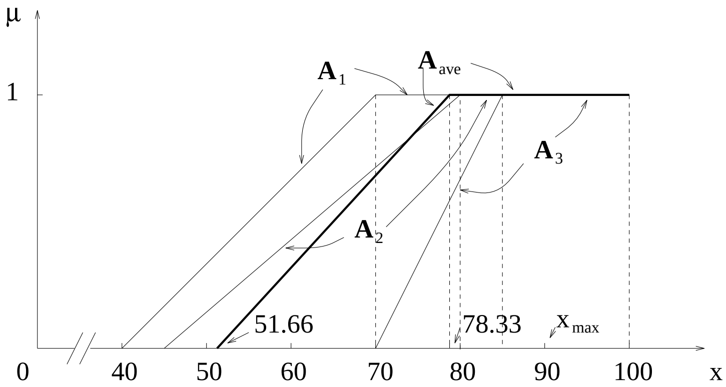


Fig. 4.14 Investment planning policy: three close experts opinions; aggregated decision $\mathcal{A}_{ave}$; maximizing decision x_{max}.

Defuzzification of $\mathbf{A}_{ave}$ using (3.17) gives the maximizing value $\frac{78.33+100}{2} = 89.16 \approx 90$. The interpretation of this number is *very aggressive* investment policy.

Assume now that the three experts are evaluated differently by their peers on a scale from 0 to 10 as follows: $r_1 = 6$ is the ranking of expert $E_1, r_2 = 10$ is the ranking of expert E_2, and $r_3 = 4$ is the ranking of expert E_3. The weights $w_i, i = 1, 2, 3$, which express the relative importance of E_i can be calculated from (3.3):

$$
w_i = \frac{r_i}{r_1 + r_2 + r_3}; \ w_1 = \frac{6}{20} = 0.3; \ w_2 = \frac{10}{20} = 0.5, \ w_3 = \frac{4}{20} = 0.2.
$$

Substituting these values into the weighted trapezoidal average formula (3.14) gives

$$\begin{aligned}\mathbf{A}^w_{ave} &= 0.3\mathbf{A}_1 + 0.5\mathbf{A}_2 + 0.2\mathbf{A}_3\\ &= (12, 21, 30, 30) + (22.5, 40, 50, 50) + (14, 17, 20, 20)\\ &= (43.5, 78, 100, 100).\end{aligned}$$

Using again (3.17) for defuzzification gives $\frac{78+100}{2} = 89$; this number suggests *very aggressive* investment policy.

There is a little difference between $\mathbf{A}_{ave}$ and $\mathbf{A}^w_{ave}$ and also between the maximized (defuzzified) values 89.16 and 89. Hence the ranking of the experts in this case has no significance on the final conclusion. This is mainly due to the fact that the experts opinions are more or less close and also to the fact that the second expert E_2 which opinion is closest to $\mathbf{A}_{ave}$ was ranked as the best ($r_2 = 10$).

If the interest rates are not falling but raising the same methodology can be applied.

□

Case Study 15 *Investment Model Under Conflicting Experts Opinions*

Consider the investment model studied in Case Study 14 when interest rates are falling but assume now that the experts have conflicting opinions.[8] This means that some experts are reccommending *aggressive* policy (scale from 0 to 100) while at the same time others are reccommending *conservative* policy (scale from -100 to 0); also there is a possibility that some experts may express opinions which are almost in the middle between aggressive and conservative policy.

Suppose that three experts present their opinions on the matter (they are of equal importance) by the fuzzy numbers (see Fig. 4.15):

$$\begin{aligned}\mathbf{A}_1 &= (-100, -100, -50, -30),\\ \mathbf{A}_2 &= (-10, 10, 30),\\ \mathbf{A}_3 &= (60, 90, 100, 100);\end{aligned}$$

$\mathbf{A}_1$ (describing *conservative*) is a right trapezoidal number, $\mathbf{A}_2$ (describing *slightly aggressive*) is a triangular number, and $\mathbf{A}_3$ (describing *aggressive*) is a left trapeziodal number.

To use (3.13) for aggregation of the three conflicting opinions expressed by $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{A}_3$, first $\mathbf{A}_2$ must be presented as a trapezoidal number, $\mathbf{A}_2 = (-10, 10, 10, 30)$ (Section 3.2). The result is (Fig. 4.15)

$$\begin{aligned} & \mathbf{A}_{ave} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 \\ = & \frac{(-100, -100, -50, -30) + (-10, 10, 10, 30) + (60, 90, 100, 100)}{3} \\ = & \frac{(-50, 0, 60, 100)}{3} = (-16.67, 0, 20, 33.33). \end{aligned}$$

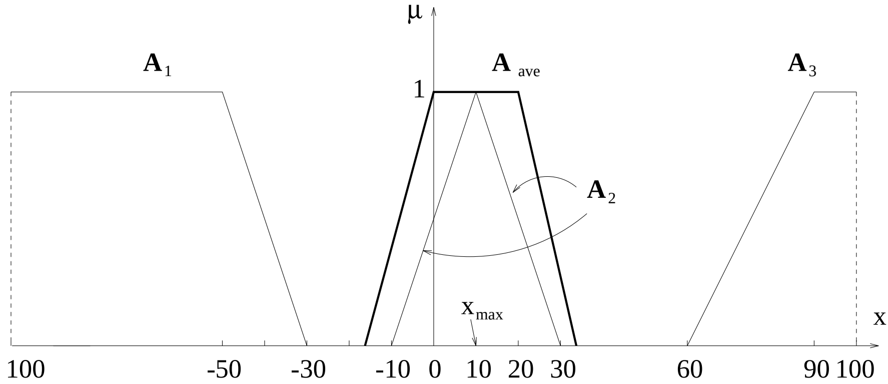


Fig. 4.15. Investment planning policy: three conflicting experts opinions; aggragated decision A_{ave}; maximizing decision x_{max}.

The maximizing value according to (3.17) is $\frac{0+20}{2} = 10$. It suggests a policy on the aggressive side of the scale but a very caustious one.

Now consider the case when the opinions of the three conflicting experts have different importance on a scale from 0 to 10. The ranking of experts $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ is assumed to be 4, 6, and 10, correspondingly. The weights w_i for $\mathbf{E}_i$ calculated from (3.3) are

$$w_i = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}; \; w_1 = \frac{4}{20} = 0.2, \; w_2 = \frac{6}{20} = 0.3, \; w_3 = \frac{10}{20} = 0.5.$$

Using (3.14) to aggragate the conflicting experts opinions gives

$$\begin{aligned} \mathbf{A}_{ave}^{w} &= 0.2\mathbf{A}_1 + 0.3\mathbf{A}_2 + 0.5\mathbf{A}_3 \\ &= (-20, -20, -10, -6) + (-3, 3, 3, 9) + (30, 45, 50, 50) \\ &= (7, 28, 43, 53) \end{aligned}$$

whose maximizing value (3.15) is $x_{max} = \frac{28+43}{2} = 35.54$. It indicates that the investment policy should be cautiously aggressive.

There is some difference between $\mathbf{A}_{ave}$ and $\mathbf{A}_{ave}^{w}$ and also between the defuzzified values 10 and 35.5 due to the high ranking of expert $\mathbf{E}_3$ who favors aggressive investment policy.

□

4.6 Fuzzy Zero-Based Budgeting

Government agencies and companies often use the *zero-based budgeting* method for budget planning with crisp data. Since the available information is usually imprecise and ambiguous, it is more realistic to use fuzzy data instead of crisp data. This is the justification for the establishment of a more general method known as *fuzzy zero-based budgeting* (Kaufmann and Gupta (1988)).

The fuzzy zero-based budgeting method uses triangular numbers to model fuzziness in budgeting. It is a decision-making procedure different from the two methods discussed in this chapter, decision making by intersection and fuzzy averaging. Since fuzzy zero-based budgeting uses addition of triangular numbers, from this point of view it is close to fuzzy averaging. It will be illustrated on a particular situation.

Consider a company with several decision centers, say A, B, and C. Assume that the decision makers agree on some preliminary budgets using a specified number of budget levels for each center depending on its importance. The budgets are expressed in terms of triangular fuzzy numbers obtained by certain procedure (it might be the Fuzzy Delphi method or some other way).

The following possible budget levels were suggested:

$$\begin{aligned}
&\text{for the center} A, \mathbf{A}_0 < \mathbf{A}_1 < \mathbf{A}_2,\\
&\text{for the center} B, \mathbf{B}_0 < \mathbf{B}_1,\\
&\text{for the center} C, \mathbf{C}_0 < \mathbf{C}_1 < \mathbf{C}_2.
\end{aligned}$$

They are schematically presented in Table 4.3.

Table 4.3 Suggested budgets for three centers.

level 2	$\mathbf{A}_2$		$\mathbf{C}_2$
level 1	$\mathbf{A}_1$	$\mathbf{B}_1$	$\mathbf{C}_1$
level 0	$\mathbf{A}_0$	$\mathbf{B}_0$	$\mathbf{C}_0$
center	A	B	C

The budget with a subscript zero (level 0) represents a minimal budget; if a center is given this budget, it might be closed. Budgets with subscript one (level 1) are normal budgets; those with subscript two or greater than two (level 2 or higher levels if such exist) are improved.

The total budget available to the company is limited but it is flexible and could be expressed by a right trapezoidal number **L** of the type shown in Fig. 4.16 with membership function

$$\mu_L(x) = \begin{cases} 1 & \text{for} \qquad 0 < x \le l_1, \\ \frac{x-l_2}{l_1-l_2} & \text{for} \qquad l_1 \le x \le l_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

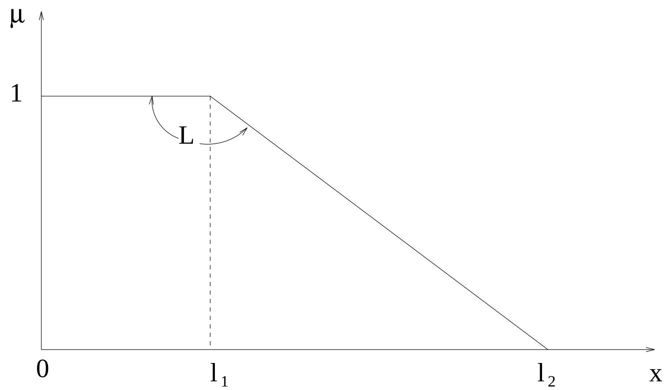


Fig. 4.16. Total available budget.

The decision makers follow a step by step budget allocation procedure according to the importance of each center in their opinion. They select a budget for a center beginning at zero level and continue until all budgets on Table 4.3 are specified. A budget on a higher level includes that on a lower level for the same center. The procedure is shown in Table 4.4; the selected budgets are presented by shaded area. From the table we see that first (Step 1) an initial budget $\mathbf{C}_0$ is allocated to the center C considered to be the most important. After that (Step 2) the center A gets support $\mathbf{A}_0$. Then again (Step 3) the center C is chosen; its budget is increased from $\mathbf{C}_0$ to $\mathbf{C}_1$ before even center B to be selected. Clearly center B is the last priority. The selection procedure continuous (Table 4.4). Step 7 for instance indicates that while centers C and A are selected for allocation at level 2 the center B is given budget on level 0; only in the last Step 8 this center gets budget on level 1.

The cumulative budgets according to Table 4.4 after dropping the lower level budgets from any center when a budget on higher level is selected, listed step by step are:

$$\begin{aligned}
\mathbf{S}_1 &= \mathbf{C}_0, \\
\mathbf{S}_2 &= \mathbf{A}_0 + \mathbf{C}_0, \\
\mathbf{S}_3 &= \mathbf{A}_0 + \mathbf{C}_1, \\
\mathbf{S}_4 &= \mathbf{A}_0 + \mathbf{C}_2, \\
\mathbf{S}_5 &= \mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_2, \\
\mathbf{S}_6 &= \mathbf{A}_1 + \mathbf{B}_0 + \mathbf{C}_2, \\
\mathbf{S}_7 &= \mathbf{A}_2 + \mathbf{B}_0 + \mathbf{C}_2, \\
\mathbf{S}_8 &= \mathbf{A}_2 + \mathbf{B}_1 + \mathbf{C}_2.
\end{aligned} \tag{4.13}$$

The budgets $\mathbf{S}_i, i = 1, \ldots, 8$ are triangular numbers since they are sums of triangular numbers (Section 3.2 (3.4)). They can be presented in the form $\mathbf{S}_i = (s_1^{(i)}, s_M^{(i)}, s_2^{(i)})$.

The final budget has to be selected from (4.13). The company wants to have an optimal fuzzy budget $\mathbf{S}_{opt} = (s_1, s_M, s_2)$ with peak $(s_M, 1)$ consistent with the available budget $\mathbf{L}$. Hence it is reasonable and prudent to require

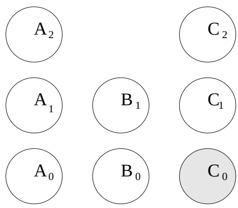
$$\mathbf{S}_{opt} = (s_1, s_M, s_2) \subseteq \mathbf{L}, \tag{4.14}$$

where

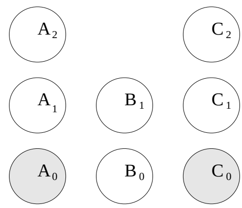
$$s_M = \max s_M^{(i)} \le l_1, \qquad s_2 = \max s_2^{(i)} \le l_2, \tag{4.15}$$

i.e. s_M is the largest $s_M^{(i)} \le l_1$ and s_2 is the largest $s_2^{(i)} \le l_2, i = 1, \ldots, 8$ (see Fig. 4.16 for l_1 and l_2).

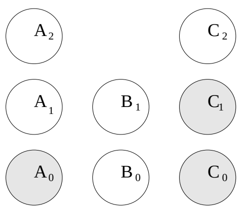
Table 4.4. Procedure for budget selection.



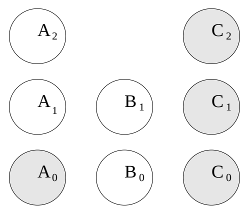
Step 1, $\mathbf{C}_0$ is selected



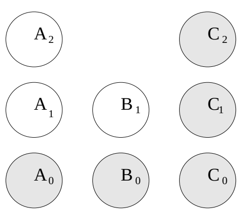
Step 2, $\mathbf{A}_0$ is selected



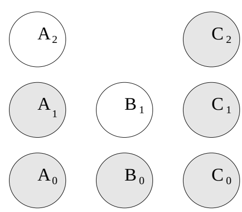
Step 3, $\mathbf{C}_1$ is selected



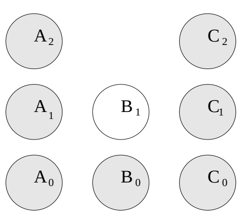
Step 4, $\mathbf{C}_2$ is selected



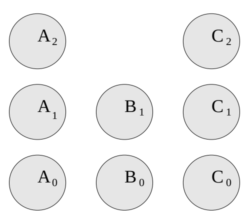
Step 5, $\mathbf{B}_0$ is selected



Step 6, $\mathbf{A}_1$ is selected



Step 7, $\mathbf{A}_2$ is selected



Step 8, $\mathbf{B}_1$ is selected

The inclusion (4.14) interpreted as a requirement that the budget $\mathbf{S}_{opt}$ does not exceed the available budget $\mathbf{L}$ essentially means that $\mathbf{S}_{opt}$ entails $\mathbf{L}$ (see Section 2.7 (2.14)).

If a crisp budget is needed, the company could take as such the maximizing value (see (3.15)) $x_{\max} = s_M$ in (4.14).

Condition (4.14) with (4.15) is suitable for a conservative budget. A company expecting additional funding which may or may not materialize or willing to take risk may decide to relax the inclusion (4.14) and substitute it with

$$\mathbf{S}_{opt} \approx \mathbf{L}.$$

In such a case the first condition (4.15) is required, the second is dropped or vise versa, or both conditions (4.15) are dropped but substituted instead by $s_M = \min s_M^{(i)} > l_1$.

Case Study 16 *Application of Fuzzy Zero-Based Budgeting*

Let us assign specified values to the fuzzy numbers in the particular situation considered above.

The limited available budget $\mathbf{L}$ (see (4.12)) given by

$$\mu_L(x) = \begin{cases} 1 & \text{for} \quad 0 < x \leq 40000, \\ -\frac{x-46000}{6000} & \text{for} \quad 40000 \leq x \leq 46000, \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

is shown in Fig. 4.17 and the eight budgets on Table 4.3 are selected as follows

$$\begin{aligned} \mathbf{A}_0 &= (10000, 11000, 12000), \\ \mathbf{A}_1 &= (12000, 13000, 15000), \\ \mathbf{A}_2 &= (14000, 15000, 17000), \\ \mathbf{B}_0 &= (7000, 9000, 11000), \\ \mathbf{B}_1 &= (11000, 12000, 13000), \\ \mathbf{C}_0 &= (7000, 9000, 12000), \\ \mathbf{C}_1 &= (11000, 13000, 15000), \\ \mathbf{C}_2 &= (15000, 18000, 19000). \end{aligned}$$

For the cumulative budgets (4.13) using addition of triangular fuzzy numbers (Section 3.2) we find

$$\begin{aligned}
\mathbf{S}_1 &= \mathbf{C}_0 = (7000, 9000, 12000),\\
\mathbf{S}_2 &= \mathbf{A}_0 + \mathbf{C}_0 = (17000, 20000, 24000),\\
\mathbf{S}_3 &= \mathbf{A}_0 + \mathbf{C}_1 = (21000, 24000, 27000),\\
\mathbf{S}_4 &= \mathbf{A}_0 + \mathbf{C}_2 = (25000, 29000, 31000),\\
\mathbf{S}_5 &= \mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_2 = (32000, 38000, 42000),\\
\mathbf{S}_6 &= \mathbf{A}_1 + \mathbf{B}_0 + \mathbf{C}_2 = (34000, 40000, 45000),\\
\mathbf{S}_7 &= \mathbf{A}_2 + \mathbf{B}_0 + \mathbf{C}_2 = (36000, 42000, 47000),\\
\mathbf{S}_8 &= \mathbf{A}_2 + \mathbf{B}_1 + \mathbf{C}_2 = (39000, 45000, 49000).
\end{aligned}$$

The budgets $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$, and $\mathbf{S}_4$ are too small in comparison to the limiting budget $\mathbf{L}$. Hence the company discards them and considers the rest, $\mathbf{S}_5, \mathbf{S}_6, \mathbf{S}_7$, and $\mathbf{S}_8$ shown in Fig. 4.17 together with $\mathbf{L}$. However the budgets $\mathbf{S}_7$ and $\mathbf{S}_8$ violate condition (4.14).

The budgets $\mathbf{S}_5$ and $\mathbf{S}_6$ have a peak 1 for $s_M^{(5)} = 38000$ and $s_M^{(6)} = 40000$, correspondingly, but since $s_M^{(5)} < s_M^{(6)} = l_1 = 4000$ and $s_2^{(5)} < s_2^{(6)} < l_2 = 4600$, the optimal budget (see (4.14) and (4.15)) is $\mathbf{S}_6 = (34000, 40000, 45000)$ and the crisp budget is $x_{\max} = s_M^{(6)} = 40000$. If the company accepts this budget, recalling that $\mathbf{S}_6 = \mathbf{A}_1 + \mathbf{B}_0 + \mathbf{C}_2$, the center A gets budget $\mathbf{A}_1$ (crisp 13000), the center B gets budget $\mathbf{B}_0$ (crisp 9000), and the center C gets budget $\mathbf{C}_2$ (crisp 18000).

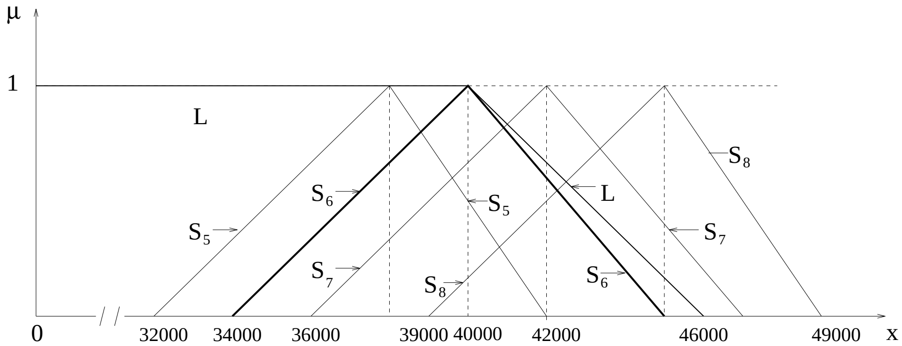


Fig. 4.17. Cumulative budgets.

The budget of center B is at level 0 (smaller than normal); the decision makers may consider the option to close this center and redistribute the money to the other two centers which are more important.

If the company management wants to be more flexible and have reasons to be more optimistic, then the budget $\mathbf{S}_7 = (36000, 42000, 47000)$ could be considered (crisp 42000). This budget satisfies the condition that $s_M^{(7)}$ is the smallest $s_M^{(i)} > l_1 = 4000$.

□

4.7 Notes

1. According to Nuala Beck (1995) "the skills that all of us need to get ahead in this challenging times" are: "the ability to work as part of a team, ... the ability to communicate, ... the ability to use a computer, ... the ability to do basic math."

   Nuala Beck in her book (1992) on the new economy writes: "Artificial intelligence and fuzzy logic systems, already in use experimentally in insurance and banking and defense, will find their way indo education ...." "Each era has its *winners* and *losers*. It's not too early to predict that the losers of tomorrow will include many of winners of today. If a successful company starts believing it has all the answers—or that its tree will grow to the sky—it is already heading down the wrong track. If a Microsoft, for example, doesn't go beyond software and make the leap into artificial intelligence and commercialize fuzzy logic on a massive scale, then its star will inevitably fall."

2. The idea for Case Study 6 comes from Novák (1989).

3. The specific data concerning job selection by Mary (Case Study 8) are modification of data given by Klir and Folger (1988).

4. Case Study 9 is based on material in the book by Li and Yen (1995).

5. Some of the requirements (rules) concerning pricing of new products (Section 4.3) are based on Cox (1995); the linguistic values in his book are described by bell-shaped fuzzy numbers.

6. Grant (1993) in the chapter on assessing profit prospects in his book writes: "To survive and prosper in the face of price competition requires that the firm establishes a low-cost position."

7. One of the five deadly business sins according to Drucker (Managing in a Time of Great Change, 1995) is "cost-driven pricing." Further he writes: "The only thing that works is price-driven costing. Most American and practically all European companies arrive at their prices by adding up costs and putting a profit margin on top .... Their argument? We have to recover our costs and make a profit. This is true but irrelevant: customers do not see it as their job to ensure manufacturers a profit ... Cost-driven pricing is the reason there is no American consumer-electronics industry anymore. It had the technology and the products. But it operated on cost-led pricing—and the Japanese practiced price-led costing."

8. Case studies 14 and 15 in Section 4.5 deal with individual planning policy wihch depends on falling or rising prime interest rates. This reflects only one facet of the problem. The experts also should relay on data concerning the state of the stock market, the trade balance, unemployment rate, level of inventory stockage, etc. In that connection, and to stress the complexity of that type of problems in business and finance where many factors are involved and interrelated, and also to focus on a moral issue, we make a quote from the article "Wanted, Economic Vision That Focuses on Working People" by B. Herbert (*International Herald Tribune*, July 10, 1996). "Last Friday, a kernel of good news on the employment front caused a panic on Wall Street. The consensus: The Fed will have to raise interest rates to ensure that any improvement do not get out of hand."